Main Theorems in SLP regarding Dynamic Programming

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Disclaimer

This is a summary of "main" definitions and theorems from Stokey and Lucas with Prescott (1989), intended for study purposes. No original content is claimed by the author of these notes.

Chapter 3 *Mathematical Preliminaries*

Definition *Let (S, ρ) be metric space and $T : S \to S$ be a function mapping S into itself. T is a* ***contraction mapping*** *with modulus β if for some $\beta \in (0, 1)$ it holds $\rho(Tx, Ty) < \beta\rho(x, y)$ $\forall x, y \in S$.*

Theorem 3.2 **(Contraction Mapping Theorem)** *If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then*
a. T has exactly one fixed point v in S and
b. for any $v_0 \in S$, $\rho(T^n v, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2, \ldots$

Corollary 1 Let (S, ρ) be a metric space, and let $T : S \to S$ be a contraction mapping with a fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then $v \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$.

Theorem 3.3 **(Blackwell's sufficient conditions for a contraction)** *Let $x \subseteq \mathbf{R}^l$, and let $B(X)$ be a space bounded functions $f : X \to \mathbf{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying*
a. (monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, $\forall x \in X$ implies $(Tf)(x) \leq (Tg)(x)$, $\forall x \in X$
b. (discounting) there exists some $\beta \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a \qquad \text{all } f \in B(X), a \geq 0, x \in X$$

Then T is a contraction with modulus β. (Here by definition $(f + a)(x) = f(x) + a$)

Definition *A correspondence $\Gamma : X \to Y$ is* ***lower hemi-countinuous*** *(l.h.c) at $x \in X$ if $\Gamma(x)$ is nonempty and if, for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$, all $n \geq N$. If $\Gamma(\cdot)$ is nonempty-valued on X, then it is always possible to take $N = 1$.*

$\Gamma : X \to Y$ is l.h.c. on X if $\forall x \in X$ $\Gamma(x) \neq \emptyset$ and $\forall y \in \Gamma(x)$ $\forall \{x_n\}$ such that $x_n \to x$ $\exists N \geq 1$ $\exists \{y_n\}_{n=N}^{\infty}$ such that $y_n \in \Gamma(x_n), \forall n \geq N$ and $y_n \to y$

Definition *A compact-valued correspondence $\Gamma : X \to Y$ is* ***upper hemi-countinuous*** *(u.h.c) at $x \in X$ if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$, and every sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$, all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point is in $\Gamma(x)$.*

$\Gamma : X \to Y$ is u.h.c. on X if $\forall x \in X$ $\Gamma(x) \neq \emptyset$ and $\forall \{x_n\}$ such that $x_n \to x$ $\forall \{y_n\}$ such that $\forall n$ $y_n \in \Gamma(x_n)$ $\exists \{y_{n_k}\}$ such that $y_{n_k} \to y$, $y \in \Gamma(x)$

Theorem 3.4 *Let $\Gamma : X \to Y$ be a nonempty-valued correspondence, and let A be the graph of Γ. Suppose that A is closed, and that for any bounded set $\hat{X} \subseteq X$, the set $\Gamma(\hat{X})$ is bounded. Then Γ is compact-valued and u.h.c.*

Theorem 3.6 *(***Theorem of Maximum)** *Let $X \subseteq \mathbf{R}^l$ and $Y \subseteq \mathbf{R}^m$, let $f : X \times Y \to R$ be a continuous function, and let $\Gamma : X - \to Y$ be a compact-valued and continuous correspondence. Then the function $h : X \to \mathbf{R}$, $h(x) = max_{y \in \Gamma(x)} f(x,y)$is continuous, and the correspondence $G : X \to Y$, $G(x) = \{y \in \Gamma(x) : f(x,y) = h(x)\}$ is nonempty-valued, compact-valued and u.h.c.*

Chapter 4 *Dynamic Programming under Certainty*

Let X be a subset of an Euclidean space, and let

$$\begin{aligned}
&\Gamma : X \to X \\
&A = \{(x,y) \in X \times X : y \in \Gamma(x)\} \\
&F : A \to \mathbf{R} \\
&\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t),\ t = 0, 2, \ldots\}
\end{aligned}$$

Sequential problem

$$\text{(SP)} \qquad \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t} \quad x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \ldots$$
$$x_0 \in X \text{ given}$$

Functional equation

$$\text{(FE)} \qquad v(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)], \qquad \text{all } x \in X$$

Define $u_n : \Pi(x_0) \to \mathbf{R}$, $u : \Pi(x_0) \to \mathbf{R}$, $v^* : X \to \mathbf{R}$ as

$$\begin{aligned}
u_n(\underset{\sim}{x}) &= \sum_{t=1}^{n} \beta^n F(x_t, x_{t+1}) \\
u(\underset{\sim}{x}) &= \lim_{n \to \infty} u_n(\underset{\sim}{x}) \\
v^*(x_0) &= \sup_{\underset{\sim}{x} \in \Pi(x_0)} u(\underset{\sim}{x})
\end{aligned}$$

Assumption 4.1 *$\Gamma(x)$ is nonempty, for all $x \in X$*

Assumption 4.2 *For all $x_0 \in X$ and $\underset{\sim}{x}\Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists in $\bar{\mathbf{R}}$.*

Assumption 4.3 *X is a convex subset of $\mathbf{R}^l$, and the correspondence $\Gamma : X \to X$ is nonempty, compact-valued and continuous.*

Assumption 4.4 *The function $F : A \to \mathbf{R}$ is bounded and continuous, and $0 < \beta < 1$.*

Assumption 4.5 *For each y, $F(x, y)$ is a strictly increasing in each of its first l arguments.*

Assumption 4.6 *Γ is monotone in the sense that $x \le x'$ implies $\Gamma(x) \subseteq \Gamma(x')$.*

Assumption 4.7 *F is strictly concave, that is*

$$F(\theta(x, y) + (1-\theta)(x', y')) \ge \theta F(x, y) + (1-\theta)F(x', y'), \quad \text{all } (x, y), (x'y') \in A, \text{ and all } \theta \in (0, 1)$$

and the inequality is strict is $x \neq x'$.

Assumption 4.8 *Γ is convex in the sense that for any $\theta \in (0, 1)$, and any $x, x' \in X$ $y \in \Gamma(x)$, $y' \in \Gamma(x')$ implies $\theta y + (1-\theta)y' \in \Gamma(\theta x + (1-\theta)x')$.*

Assumption 4.9 *F is continuously differentiable on the interior of A.*

Theorem 4.2 *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. Then the function v^* satisfies (FE).*

Theorem 4.3 *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. If v is the solution to (FE) and satisfies*

$$\lim_{n\to\infty} \beta^n v(x_n) = 0, \text{ all } (x_0, x_1, \ldots) \in \Pi(x_0), \text{ all } x_0 \in X$$

then $v = v^*$.

Theorem 4.4 *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. Let $\underset{\sim}{x}^* \in \Pi(x_0)$ be a feasible plan that attains the supremum in (SP) for initial state x_0. Then*

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v(x_{t+1}^*), \qquad t = 0, 1, 2, \ldots$$

Theorem 4.5 *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. Let $\underset{\sim}{x}^* \in \Pi(x_0)$ be a feasible plan from x_0 satisfying*

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v(x_{t+1}^*), \qquad t = 0, 1, 2, \ldots$$

and with

$$\limsup_{t\to\infty} \beta^t v^*(x_t^*) \le 0.$$

Then $\underset{\sim}{x}^*$ attains the supremum in (SP) for initial state x_0.

Theorem 4.6 *Let X, Γ, F and β satisfy Assumptions 4.3-4.4, and let $C(X)$ be the space of bounded continuous functions $f : X \to \mathbf{R}$, with the supnorm $||f|| = \sup_{x\in X} |f(x)|$. Then the operator T on $C(X)$ defined by*

$$(Tf)(x) = \max_{y\in\Gamma(x)} [F(x, y) + \beta f(y)]$$

maps $C(X)$ into itself, $T : C(X) \to C(X)$; it has a unique fixed point $v \in C(X)$; and for all $v_0 \in C(X)$,

$$||T^n v_0 - v|| \le \beta^n ||v_0 - v||, \quad n = 0, 1, 2, \ldots$$

Moreover given v the optimal policy correspondence $G : X \to X$ defined by

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

is compact-valued and u.h.c.

Theorem 4.7 *Let X, Γ, F and β satisfy Assumptions 4.3-4.6, and let v be a unique solution to*

$$v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

then v is strictly increasing.

Theorem 4.8 *Let X, Γ, F and β satisfy Assumptions 4.3-4.4, and 4.7-4.8, let v be a unique solution to*

$$v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

and G satisfy

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

then v is strictly concave and G is a continuous, single-valued function.

Theorem 4.11 *Let X, Γ, F and β satisfy Assumptions 4.3-4.4, and 4.7-4.9, let v and g satisfy*

$$v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

and

$$g(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

If $x_0 \in intX$ and $g(x_0) \in \Gamma(x_0)$, then v is continuously differentiable at x_0, with derivatives given by $v_i(x_0) = F_i(x_0, g(x_0))$, $i = 1, 2, \ldots, l$.

Constant Returns to Scale

Assumption 4.10 *$X \subseteq R^l$ is a convex cone. The correspondence $\Gamma : X \to X$ is nonempty, compact-valued, and continuous; for any $x \in X$,*

$$y \in \Gamma(x) \text{ implies } \lambda y \in \Gamma(\lambda x), \text{ all } \lambda > 0$$

that is, graph of Γ is cone. In addition, for some $\alpha \in (0, \beta^{-1})$,

$$||y||_l \leq \alpha ||x||_l, \text{ all } y \in \Gamma(x), \text{ all } x \in X$$

Assumption 4.11 *The function $F : A \to \mathbf{R}$ is continuous and homogeneous of degree one, and for some $0 < B < \infty$*

$$|F(x, y)| \leq B(||x||_l + ||y||_l), \text{ all } (x, y) \in A$$

and $\beta \in (0, 1)$.

Exercise 4.6 *Under Assumptions 4.10-4.11 $||x_t||_l \leq \alpha^t ||x_0||_l$, for all $x_0 \in X$, all $\{x_t\} \in \Pi(x_0)$, Assumptions 4.1-4.2 hold, and the supremum function $v^* : X \to \mathbf{R}$ is homogeneous of degree one, and for some $0 < c < \infty$ satisfies $|v^*(x)| \leq c||x||_l$, all $x \in X$.*

Exercise 4.7 *Let $H(X)$ be the space of functions $f : X \to \mathbf{R}$ that are continuous and homogeneous of degree one, and bounded in the norm*

$$||f|| = \max_{||x||=1, x \in X} |f(x)|$$

Then $H(X)$ is a Banach space (a complete normed vector space) and under Assumptions 4.10-4.11 operator

$$(Tf)(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)]$$

maps $H(X)$ into itself, that is $T : H(X) \to H(X)$.

For any f homogeneous of degree 1, and any $a \in \mathbf{R}$ define $f + a$ by

$$(f + a)(x) = f(x) + a||x||$$

Since f is homogeneous of degree one and $||\lambda x|| = \lambda||x||$ for all $\lambda > 0$, also $f + a$ is homogeneous of degree one.

Theorem 4.12 *Let X be a convex cone, $H(X)$ be a space of continuous, homogeneous of degree 1 functions, bounded in the norm $\max_{||x||=1, x\in X} |f(x)|$, and let the mapping $T : H(X) \to H(X)$ satisfy*
a. monotonicity: for any $f, g \in H(X)$, if $f \leq g$ then $Tf \leq Tg$
b. discounting: exists $\gamma \in (0, 1)$ such that for all $f \in H$ and all $a > 0$, $T(f + a) \leq Tf + \gamma a$
Then T is a contraction with modulus γ.

Theorem 4.13 *Let X, Γ, F and β satisfy Assumptions 4.10 and 4.11, and let $H(X)$ be a space of continuous, homogeneous of degree 1 functions, bounded in the norm $\max_{||x||=1, x\in X} |f(x)|$. Then operator T on $H(X)$ defined by*

$$(Tf)(x) = \sup_{y\in\Gamma(x)} [F(x, y) + \beta v(y)]$$

maps $H(X)$ into itself, that is $T : H(X) \to H(X)$, and has a unique fixed point $v^* \in H(x)$. In addition

$$||T^n v_0 - v^*|| \leq (\alpha\beta)^n ||v_0 - v^*||, \quad n = 0, 1, 2, \dots, \text{ all } v_0 \in H(X)$$

and the associated policu correspondence $G : X \to X$ is a compact valued and u.h.c. Moreover, G is homogeneous of degree one, for any $x \in X$

$$y \in G(x) \text{ implies } \lambda y \in G(\lambda x), \text{ all } \lambda \geq 0$$

Chapter 7 *Measure Theory and Integration*

Definition *Let S be a set and let $\mathscr{S}$ be a family of subsets of S. $\mathscr{S}$ is called **σ-algebra** if*
a. $\emptyset, S \in \mathscr{S}$;
b. $A \in \mathscr{S}$ implies $A^c = S\backslash A \in \mathscr{S}$; and
*c. $A_n \in \mathscr{S}$, $n = 1, 2, \dots$, implies $\bigcup_{n=1}^{\infty} A_n \in \mathscr{S}$. A pair $(S, \mathscr{S})$ where S is a set and $\mathscr{S}$ is a σ-algebra of its subsets is called a **measurable space**. Any set $A \in \mathscr{S}$ is called an $\mathscr{S}$-**measurable set**.*

Definition *The smallest σ-algebra containing all the open sets of the form $(-\infty, b)$, (a, b), $(a, +\infty)$, $(-\infty, +\infty)$ is called a **Borel algebra** for $\mathbf{R}^1$ and is denoted by $\mathscr{B}^1$, any set in $\mathscr{B}^1$ is called a **Borel set**.*
In general, for any metric space (S, ρ) Borel algebra is the smallest σ-algebra containing all the open sets of the form $A = \{s \in S : \rho(s, s_0) < \delta$ where $s_0 \in S$, $\delta > 0$. For $S = \mathbf{R}^l$ with Euclidean metric the Borel algebra is denoted $\mathscr{B}^l$.
For any borel set $S \subset \mathbf{R}^l$ we define $\mathscr{B}_S = \{A \in \mathscr{B}^l : A \subseteq S\}$ to be the borel subsets of S.

Definition *Let $(S, \mathscr{S})$ be a measurable space. A **measure** is an extended real-valued function $\mu : \mathscr{S} \to \bar{\mathbf{R}}$ such that*
a. $\mu(\emptyset) = 0$;
b. $\mu(A) \geq 0$, all $A \in \mathscr{S}$;
c. if $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence of subsets in $\mathscr{S}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Definition *A **measure space** is a triple $(S, \mathscr{S}, \mu)$ where S is a set, $\mathscr{S}$ is a σ-algebra of its subsets, and μ is a measure defined on $\mathscr{S}$.*

Definition *If $(S, \mathscr{S})$ is a measurable space with measure $\mu : \mathscr{S} \to \bar{\mathbf{R}}$ and in addition $\mu(S) = 1$, then μ is called a **probability measure** and $(S, \mathscr{S}, \mu)$ is a **probability space**.*

Definition *Given a measurable space $(S, \mathscr{S})$, a real-valued function $f : S \to \mathbf{R}$ is $\mathscr{S}$-**measurable** if $\{s \in S : f(s) \leq a\} \in \mathscr{S}$, for all $a \in \mathbf{R}$. Denote by $M(S, \mathscr{S})$ the space of measurable, extended real-valued functions on S, and by $M^+(S, \mathscr{S})$ the subset of nonnegative functions.*

Theorem 7.4 (Pointwise convergence preserves measurability) *Let $(S, \mathscr{S})$ and $(T, \mathscr{T})$ be measurable spaces, and let $\{f_n\}$ be a sequence of $\mathscr{S}$-measurable functions converging pointwise to f i.e. $\lim f_n(s) = f(s)$, all $s \in S$. Then f is also $\mathscr{S}$-measurable.*

Definition *Let $(S, \mathscr{S})$ and $(T, \mathscr{T})$ be measurable spaces. Then the funciton $f : S \to T$ is **measurable** if the inverse image of every measurable set is measurable, that is if $\{s \in S : f(s) \in A\} \in \mathscr{S}$, for all $A \in \mathscr{T}$.*

Definition *Let $(S, \mathscr{S})$ and $(T, \mathscr{T})$ be measurable spaces, and let Γ be a correspondence of S into T. Then the function $h : S \to T$ is a **measurable selection from Γ** if h is measurable and $h(s) \in \Gamma(s)$, all $s \in S$.*

Theorem 7.6 (Measurable Selection Theorem) *Let $S \subseteq \mathbf{R}^l$ and $T \subseteq \mathbf{R}^m$ be Borel sets, with their Borel subsets $\mathscr{S}$, $\mathscr{T}$. Let $\Gamma : S \to T$ be a non-empty, compact-valued and u.h.c. correspondence. Then there exists a measurable selection from Γ.*

Theorem 7.8 (Monotone Convergence Theorem) *If $\{f_n\}$ is a monotone increasing sequence of functions in $M^+(S, \mathscr{S})$ converging pointwise to f then*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Definition *Let $(S, \mathscr{S}, \mu)$ be a measure space, and let f be a measurable, real-valued function on S. If f^+ and f^- both have finite integrals with respect to μ then f is **integrable**, and the integral with respect to μ is*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

If $A \in \mathscr{S}$, the integral of f over A with respect to μ is

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu$$

Theorem 7.10 (Lebesgue Dominated Convergence Theorem) *Let $(S, \mathscr{S}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of integrable functions that converges almost everywhere to a measurable function f. If there exists an integrable function g such that $|f_n| \leq g$, all n then f is integrable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Definition *Let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be measurable spaces, and let Z be the Cartesian product of X and Y*

$$Z = X \times Y = \{z = (x, y) : x \in X, y \in Y\}$$

*A set $C = A \times B \subseteq Z$ is a **measurable rectangle** if $A \in \mathscr{X}$ and $B \in \mathscr{Y}$.*

Chapter 8 *Markov Processes*

Definition *Let $(Z, \mathscr{Z})$ be a measurable space. A **transition function** is a function $Q : Z \times \mathscr{Z} \to [0, 1]$ such that*

a. *for each $z \in Z$, $Q(z, \cdot)$ is a probability measure on $(Z, \mathscr{Z})$; and*
b. *for each $A \in \mathscr{Z}$, $Q(\cdot, A)$ is a $\mathscr{Z}$-measurable function.*

For any $\mathscr{Z}$-measurable function f define Tf by

$$(Tf)(z) = \int f(z')Q(z, dz'), \quad \text{all } z \in Z$$

For any measure λ on $(Z, \mathscr{Z})$ define $T^*\lambda$ by

$$T^*\lambda(A) = \int Q(z, A)\lambda(dz), \quad \text{all } A \in \mathscr{Z}$$

Theorem 8.1 *Operator T maps the space of non-negative $\mathscr{Z}$-measurable, extended-real valued functions into itself, that is $T : M^+(Z, \mathscr{Z}) \to M^+(Z, \mathscr{Z})$. Consequently, T maps the space of bounded $\mathscr{Z}$-measurable functions into itself $T : B(Z, \mathscr{Z}) \to B(Z, \mathscr{Z})$.*

Theorem 8.2 *Operator T^* maps the space of probability measures on $(Z, \mathscr{Z})$ into itself, that is $T^* : \Lambda(Z, \mathscr{Z}) \to \Lambda(Z, \mathscr{Z})$.*

Definition *A transition function Q on $(Z, \mathscr{Z})$ has the **Feller property** if the associated operator T maps the space of bounded continuous functions on Z into itself; that is, if $T : C(Z) \to C(Z)$.*

Definition *A transition function Q on $(Z, \mathscr{Z})$ is **monotone** if the associated operator T has the property that for every nondecreasing function $f : Z \to \mathbf{R}$, the function Tf is also nondecreasing.*

Let $(Z, \mathscr{Z})$ be a measurable space and for any finite $t = 1, 2, \ldots$ let

$$(Z \times \ldots \times Z, \mathscr{Z} \times \ldots \times \mathscr{Z}), \qquad (t\text{-times})$$

denote the product space. For any rectangle $B = A_1 \times \ldots \times A_t \in \mathscr{Z}^t$ let

$$\mu^t(z_0, B) = \int_{A_1} \ldots \int_{A_t} Q(z_{t-1}, dz_t)Q(z_{t-2}, dz_{t-1}) \ldots Q(z_0, dz_1)$$

Chapter 9 *Stochastic Dynamic Programming*

The stochastic dynamic problems analyzed in this chapter have one of the following two forms

$$v(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int_Z v(y, z')Q(z, dz') \right\}$$
$$v(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int_Z v(\phi(x, y, z'), z')Q(z, dz') \right\}$$

Type 1 problems.

Functional equation has the form

$$v(x,z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int_Z v(y,z')Q(z,dz') \right\}$$

Let $(X,\mathscr{X})$, $(Z,\mathscr{Z})$ be measurable spaces, and let $(S,\mathscr{S}) = (X \times Z, \mathscr{X} \times \mathscr{Z})$ and $(Z^t, \mathscr{Z}^t)$ be product spaces. Let Q be a stationary transition function on $(Z,\mathscr{Z})$. Let

$$\begin{aligned} &\Gamma : X \times Z \to X \\ &A = \{(x,y,z) \in X \times X \times Z : y \in \Gamma(x,z)\} \\ &F : A \to \mathbf{R} \end{aligned}$$

be the correspondence that defines the restriction on possible actions, its graph and the one period returns function. Define further

$$\mathscr{A} = \{C \in \mathscr{X} \times \mathscr{X} \times \mathscr{Z} : C \subseteq A\}$$

Givens for type 1 problem are $(X,\mathscr{X})$, $(Z,\mathscr{Z})$, Q, Γ, F, β.

Definition *A* ***plan*** $\pi = (\pi_0, \pi_1, \ldots)$ *is a value* $\pi_0 \in X$ *and a sequence of measurable functions* $\pi_t : Z^t \to X$, $t = 1, 2, \ldots$.
A plan π *is* ***feasible from*** $s_0 \in S$ *if* $\pi_0 \in \Gamma(s_0)$ *and* $\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t)$*, all* $z^t \in Z^t$, $t = 1, 2, \ldots$*. Denote* $\Pi_0(s_0)$ *the set of plans that are feasible from* s_0.

Assumption 9.1 Γ *is nonempty-valued and the graph of* Γ *is* $(\mathscr{X} \times \mathscr{X} \times \mathscr{Z})$*-measurable. In addition,* Γ *has a measurable selection, that is there exists a measurable function* $h : S \to X$ *such that* $h(s) \in \Gamma(s)$*, all* $s \in S$.

Assumption 9.2 $F : A \to \mathbf{R}$ *is* $\mathscr{A}$*-measurable and either (a) or (b) holds*

a. $F \geq 0$ *of* $F \leq 0$
b. *For each* $(x_0, z_0) = s_0 \in S$ *and each plan* $\pi \in \Pi(s_0)$

$$F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t) \text{ is } \mu^t(z_0, \cdot)\text{-integrable}, t = 1, 2, \ldots$$

and the limit

$$F(x_0, \pi_0, z_0) + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z^t) \mu^t(z_0, dz^t)$$

exists (although it may be plus or minus infinity)

Assumption 9.3 *If* F *takes on both signs, there is a collection of nonnegative, measurable functions* $L_t : S \to \mathbf{R}_+,\ t = 0, 1, \ldots$*, such that for all* $\pi \in \Pi(s_0)$ *and all* $s_0 \in S$

$$\begin{aligned} &|F(x_0, \pi_0, z_0)| \leq L_0(s_0) \\ &|F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)| \leq L_t(s_0), \qquad \text{all } z^t \in Z^t,\ t = 1, 2, \ldots \end{aligned}$$

and

$$\sum_{t=0}^{\infty} \beta^t L_t(s_0) < \infty$$

Assumption 9.4 X *is a convex Borel set in* $\mathbf{R}^l$*, with its Borel subsets* $\mathscr{X}$.

Assumption 9.5 *One of the following conditions holds:*

a. Z is a countable set and $\mathscr{Z}$ is the σ-algebra containing all subsets of Z

b. Z is a compact Borel subset in $\mathbf{R}^k$, with its Borel subsets $\mathscr{Z}$, and the transition function Q on $(Z, \mathscr{Z})$ has the Feller property.

Note. We need that the operator T associated with Q maps the space of bounded continuous functions into itself. If Z is a countable set, we use discrete metric and all functions on Z are continuous, this requirement is vacuous. Notice that a sequence $\{s_n = (x_n, z_n)\}$ in S converges to $s = (x, z) \in S$ if and only if $x_n \to x$ and $z_n \to z$. If Z is countable set a function on S is continuous if and only if each z-section $f(\cdot, z) : X \to \mathbf{R}$ is continuous. We take $C(S)$ the space of bounded continuous functions $f : S \to \mathbf{R}$ with the sup norm $||f|| = \sup_{s\in S} |f(s)|$

Assumption 9.6 *The correspondence $\Gamma : X \times Z \to X$ is nonempty, compact-valued, and continuous.*

Note. If Z is countable Assumption 9.6 means that for each fixed z the correspondence $\Gamma(\cdot, z) : X \to X$ is nonempty, compact valued, and continuous. Similarly, Assumption 9.7 means that for each fixed z the function $F(\cdot, \cdot, z) : A_z \to \mathbf{R}$ is bounded and continuous.

Assumption 9.7 *The function $F : A \to \mathbf{R}$ is bounded and continuous, and $\beta \in (0, 1)$.*

Assumption 9.8 *For each $(y, z) \in X \times Z$, $F(\cdot, y, z) : A_{yz} \to \mathbf{R}$ is strictly increasing.*

Assumption 9.9 *For each $z \in Z$, $\Gamma(\cdot, z) : X \to X$ is increasing in the sense that $x \leq x'$ implies $\Gamma(x, z) \subseteq \Gamma(x', z)$.*

Assumption 9.10 *For each $z \in Z$, $F(\cdot, \cdot, z) : A_z \to \mathbf{R}$ is satisfies*

$$F(\theta(x, y)+(1-\theta)(x', y'), z) \geq \theta F(x, y, z)+(1-\theta)F(x', y', z), \text{ all } (x, y), (x'y') \in A_z, \text{ and all } \theta \in (0, 1)$$

and the inequality is strict is $x \neq x'$.

Assumption 9.11 *For all $z \in Z$, all $x, x' \in X$, and for any $\theta \in (0, 1)$, $y \in \Gamma(x, z)$, $y' \in \Gamma(x', z)$ implies $\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x', z)$.*

Lemma 9.1 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Γ be given. Under Assumption 9.1, $\Pi(s_0)$ is nonempty for all $s_0 \in S$.*

Under Assumption 9.1 $\mathscr{A}$ is a σ-algebra. If F is $\mathscr{A}$-measurable for any $s_0 \in S$ and any $\pi \in \Pi(s_0)$

$$F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z^t)$$

is $\mathscr{Z}^t$- measurable, for $t = 1, 2, \ldots$

Under Assumption 9.2, for any $s_0 \in S$ we can define $u_n(\cdot, s_0) : \Pi(s_0) \to \mathbf{R}$, $n = 0, 1, \ldots$ and $u(\cdot, s_0) : \Pi(s_0) \to \overline{\mathbf{R}}$ by

$$\begin{aligned} u_0(\pi, s_0) &= F(x_0, \pi_0, z_0) \\ u_n(\pi, s_0) &= F(x_0, \pi_0, z_0) + \sum_{t=1}^{n} \int_{Z^t} \beta^t F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^t(z_0, dz^t) \\ u(\pi, s_0) &= \lim_{n\to\infty} u_n(\pi, s_0) \end{aligned}$$

Under Assumption 9.1 and 9.2, the function $u(\cdot, s)$ is well defined on the nonempty set $\Pi(s)$, for each $s \in S$. We can define supremum function $v^* : S \to \overline{\mathbf{R}}$ by

$$v^*(s) = \sup_{\pi \in \Pi(s)} u(\pi, s)$$

that is it satisfies

$$\begin{aligned} &v^*(s) \geq u(\pi, s), \quad \text{all } \pi \in \Pi(s) \\ &v^*(s) = \lim_{k \to \infty} u(\pi^k, s_0), \text{ for some sequence } \{\pi^k\}_{k=1}^{\infty} \in \Pi(s) \end{aligned}$$

Consider now

$$v(s) = v(x,z) = \sup_{y \in \Gamma(x,z)} \left[F(x,y,z) + \beta \int v(y,z')Q(z,dz') \right] \qquad \text{(FE)}$$
$$G(s) = G(x,z) = \left\{ y \in \Gamma(x,z) : v(x,z) = F(x,y,z) + \beta \int v(y,z')Q(z,dz') \right\}$$

Theorem 9.2 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F and β be given. Let Assumptions 9.1, 9.2 hold, and let v^* be the supremum function defined as above. Let v be a measurable function satisfying the functional equation (FE), and such that*

$$\lim_{t \to \infty} \int_{Z^t} \beta^t v(\pi_{t-1}(z^{t-1}), z_t)\mu^t(z_0, dz^t) = 0, \qquad \textit{all } \pi \in \Pi(s_0), \ \textit{all } (x_0, z_0) = s_0 \in S$$

Let G be the correspondence defined as above, and suppose that G is nonempty and permits a measurable selection. Then $v = v^*$, and any plan π^* generated by G attains the supremum v^*.

Theorem 9.4 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F and β be given. Let Assumptions 9.1-9.3 hold, and let v^* be the supremum function defined as above. Assume that v^* is measurable and satisfies (FE), and define the correspondence G as above. Assume that G is nonempty and permits a measurable selection. Let $(x_0, z_0) = s_0 \in S$, and let $\pi^* \in \Pi(s_0)$ be a plan that attains the supremum v^* for initial s_0. Then there exists a plan π^G generated by G from s_0 such that*

$$\begin{aligned} &\pi_0^G = \pi_0^* \\ &\pi_t^G(z^t) = \pi_t^*(z^t), \qquad \mu^t(z_0, \cdot) \textit{ almost everywhere}, t = 1, 2, \ldots \end{aligned}$$

Lemma 9.5 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$ and Q satisfy Assumptions 9.4 and 9.5. If $f : X \times Z \to \mathbf{R}$ is bounded and continuous, then Mf defined by*

$$(Mf)(y,z) = \int f(y,z')Q(z,dz'), \qquad \textit{all } (y,z) \in X \times Z$$

is also, that is $M : C(S) \to C(S)$. If f is increasing (strictly increasing) in each of its first l arguments, then so is Mf; and if f is concave (strictly concave) jointly in its first l arguments, then so is Mf.

Exercise 9.7 *If $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F and β satisfy Assumptions 9.4-9.7, then Assumptions 9.1-9.3 are satisfied.*

Theorem 9.6 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F and β satisfy Assumptions 9.4-9.7, and define operator T on $C(s)$ by*

$$(Tf)(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int f(y,z')Q(z,dz') \right\}.$$

Then $T : C(S) \to C(S)$; T has a unique fixed point v in $C(S)$ and for any $v_0 \in C(S)$

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||, \qquad n = 1, 2, \ldots$$

Moreover, the correspondence $G : S \to X$ defined by

$$G(x, z) = \left\{ y \in \Gamma(x, z) : v(x, z) = F(x, y, z) + \beta \int f(y, z')Q(z, dz') \right\}$$

is nonempty, compact-valued and u.h.c.

Theorem 9.7 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F and β satisfy Assumptions 9.4-9.9, and let v be a unique fixed point of the operator defined in Theorem 9.6. Then for each $z \in Z$, $v(\cdot, z) : X \to \mathbf{R}$ is strictly increasing.*

Theorem 9.8 *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F and β satisfy Assumptions 9.4-9.7 and 9.10-9.11; let v be a unique fixed point of the operator defined in Theorem 9.6. Then for each $z \in Z$, $v(\cdot, z) : X \to \mathbf{R}$ is strictly concave and $G(\cdot, z) : X \to X$ is a continuous (single-valued) function.*

Type 2 problems.

Functional equation has the form

$$v(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int_Z v(\phi(x, y, z'), z')Q(z, dz') \right\}$$

Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $(Y, \mathscr{Y})$ be measurable spaces, and let $(S, \mathscr{S}) = (X \times Z, \mathscr{X} \times \mathscr{Z})$ and $(Z^t, \mathscr{Z}^t)$ be product spaces. Let Q be a stationary transition function on $(Z, \mathscr{Z})$. Let

$$\Gamma : X \times Z \to Y$$
$$A = \{(x, y, z) \in X \times Y \times Z : y \in \Gamma(x, z)\}$$
$$F : A \to \mathbf{R}$$

be the correspondence that defines the restriction on possible actions, its graph and the one period returns function. Let

$$D = \{(x, y) \in X \times Y : y \in \Gamma(x, z), \text{ for some } z \in Z\}$$

and let $\phi : D \times Z \to X$ be the law of motion for the state variable. Define further

$$\mathscr{A} = \{C \in \mathscr{X} \times \mathscr{Y} \times \mathscr{Z} : C \subseteq A\}$$

Givens for type 2 problem are $(X, \mathscr{X})$, $(Y, \mathscr{Y})$, $(Z, \mathscr{Z})$, Q, Γ, F, β, ϕ.

Definition *A **plan** is a sequence of functions $\pi = \{\pi_t\}_{t=0}^{\infty}$ where $\pi_t : Z^t \to Y$, $t = 1, 2, \ldots$ is $\mathscr{Z}$-measurable, $t = 1, 2, \ldots$.*
*A plan π is **feasible from** $s_0 \in S$ if $\pi_0 \in \Gamma(s_0)$ and $\pi_t(z^t) \in \Gamma(x_t^\pi(z^t), z_t)$, all $z^t \in Z^t$, $t = 1, 2, \ldots$. where the functions $x_t^\pi : Z^t \to X$, $t = 1, 2, \ldots$ are defined recursively by $x_1^\pi(z^1) = \phi(x_0, z_0, z_1)$, all $z_1 \in Z$ and $x_t^\pi(z^t) = \phi(x_{t-1}^\pi(z^{t-1}), \pi_{t-1}(z^{t-1}), z_t)$, all $z^t \in Z^t$, $t = 2, 3, \ldots$. Denote $\Pi_0(s_0)$ the set of plans that are feasible from s_0.*

Assumption 9.1' *Γ is nonempty-valued and the graph of Γ is $(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$-measurable. In addition, Γ has a measurable selection, that is there exists a measurable function $h : S \to Y$ such that $h(s) \in \Gamma(s)$, all $s \in S$. In addition, the function $\phi : D \times Z \to X$ is measurable.*

Assumption 9.2' *$F: A \to \mathbf{R}$ is $\mathscr{A}$-measurable and either (a) or (b) holds*

a. $F \geq 0$ of $F \leq 0$

b. For each $(x_0, z_0) = s_0 \in S$ and each plan $\pi \in \Pi(s_0)$

$$F(x_t^{\pi}(z^t), \pi_t(z^t), z^t) \text{ is } \mu^t(z_0, \cdot)\text{-integrable}, t = 1, 2, \ldots$$

and the limit

$$F(x_0, \pi_0, z_0) + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t F(x_t^{\pi}(z^t), \pi_t(z^t), z_t)\mu^t(z_0, dz^t)$$

exists (although it may be plus or minus infinity)

Assumption 9.3' *If F takes on both signs, there is a collection of nonnegative, measurable functions $L_t : S \to \mathbf{R}_+,\ t = 0, 1, \ldots$, such that for all $\pi \in \Pi(s_0)$ and all $s_0 \in S$*

$$|F(x_0, \pi_0, z_0)| \leq L_0(s_0)$$
$$|F(x_t^{\pi}(z^t), \pi_t(z^t), z_t)| \leq L_t(s_0), \qquad \text{all } z^t \in Z^t,\ t = 1, 2, \ldots$$

and

$$\sum_{t=0}^{\infty} \beta^t L_t(s_0) < \infty$$

Assumption 9.16 *Y is a convex Borel set in $\mathbf{R}^m$, with its Borel subsets $\mathscr{Y}$.*

Assumption 9.17 *$\phi : D \times Z \to X$ is continuous.*

Lemma 9.1' *Let $(X, \mathscr{X})$, $(Y, \mathscr{Y})$, $(Z, \mathscr{Z})$, Γ, ϕ be given. Under Assumption 9.1', $\Pi(s_0)$ is nonempty for all $s_0 \in S$.*

Under Assumption 9.1' $\mathscr{A}$ is a σ-algebra. If F is $\mathscr{A}$-measurable for any $s_0 \in S$ and any $\pi \in \Pi(s_0)$

$$F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z^t)$$

is $\mathscr{Z}^t$- measurable, for $t = 1, 2, \ldots$

Under Assumption 9.2', for any $s_0 \in S$ we can define $u_n(\cdot, s_0) : \Pi(s_0) \to \mathbf{R}$, $n = 0, 1, \ldots$ and $u(\cdot, s_0) : \Pi(s_0) \to \overline{\mathbf{R}}$ by

$$u_0(\pi, s_0) = F(x_0, \pi_0, z_0)$$
$$u_n(\pi, s_0) = F(x_0, \pi_0, z_0) + \sum_{t=1}^{n} \int_{Z^t} \beta^t F(x_t^{\pi}(z^t), \pi_t(z^t), z_t)\mu^t(z_0, dz^t)$$
$$u(\pi, s_0) = \lim_{n\to\infty} u_n(\pi, s_0)$$

Under Assumption 9.1' and 9.2', the function $u(\cdot, s)$ is well defined on the nonempty set $\Pi(s)$, for each $s \in S$. We can define supremum function $v^* : S \to \overline{\mathbf{R}}$ by

$$v^*(s) = \sup_{\pi \in \Pi(s)} u(\pi, s)$$

that is it satisfies

$$v^*(s) \geq u(\pi, s), \quad \text{all } \pi \in \Pi(s)$$
$$v^*(s) = \lim_{k\to\infty} u(\pi^k, s_0), \text{ for some sequence } \{\pi^k\}_{k=1}^{\infty} \in \Pi(s)$$

Consider now

$$v(s) = v(x,z) = \sup_{y \in \Gamma(x,z)} \left[F(x,y,z) + \beta \int v(\phi(x,y,z'),z')Q(z,dz') \right] \qquad \text{(FE)}$$

$$G(s) = G(x,z) = \left\{ y \in \Gamma(x,z) : v(x,z) = F(x,y,z) + \beta \int v(\phi(x,y,z'),z')Q(z,dz') \right\}$$

Theorem 9.2' *Let $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$, Q,Γ,F,β and ϕ be given. Let Assumptions 9.1', 9.2' hold, and let v^* be the supremum function defined as above. Let v be a measurable function satisfying the functional equation (FE), and such that*

$$\lim_{t\to\infty} \int_{Z^t} \beta^t v(x_t^\pi(z^t), z_t)\mu^t(z_0, dz^t) = 0, \qquad \textit{all } \pi \in \Pi(s_0), \ \textit{all } (x_0,z_0) = s_0 \in S$$

Let G be the correspondence defined as above, and suppose that G is nonempty and permits a measurable selection. Then $v = v^*$, and any plan π^* generated by G attains the supremum v^*.

Theorem 9.4' *Let $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$, Q,Γ,F,β and ϕ be given. Let Assumptions 9.1'-9.3' hold, and let v^* be the supremum function defined as above. Assume that v^* is measurable and satisfies (FE), and define the correspondence G as above. Assume that G is nonempty and permits a measurable selection. Let $(x_0,z_0) = s_0 \in S$, and let $\pi^* \in \Pi(s_0)$ be a plan that attains the supremum v^* for initial s_0. Then there exists a plan π^G generated by G from s_0 such that*

$$\pi_0^G = \pi_0^*$$
$$\pi_t^G(z^t) = \pi_t^*(z^t), \qquad \mu^t(z_0,\cdot) \textit{ almost everywhere}, t = 1,2,\ldots$$

Lemma 9.5' *Let $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$ and Q satisfy Assumptions 9.4-9.5 and 9.16-9.17. If $f : X \times Z \to \mathbf{R}$ is continuous, then function $h : D \times Z \to \mathbf{R}$ defined by*

$$h(z,y,z) = \int f(\phi(x,y,z'),z')Q(z,dz')$$

is also continuous.

Exercise 9.5 *If $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$, Q,Γ,F,β and ϕ satisfy Assumptions 9.4-9.7 and 9.16-9.17, then Assumptions 9.1'-9.3' are satisfied.*

Theorem 9.6' *Let $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$, Q,Γ,F,β and ϕ satisfy Assumptions 9.4-9.7 and 9.16-9.17, and define operator T on $C(s)$ by*

$$(Tf)(x,z) = \max_{y\in\Gamma(x,z)} \left\{ F(x,y,z) + \beta \int f(\phi(x,y,z')z')Q(z,dz') \right\}.$$

Then $T : C(S) \to C(S)$; T has a unique fixed point v in $C(S)$ and for any $v_0 \in C(S)$

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||, \qquad n = 1,2,\ldots$$

Moreover, the correspondence $G : S \to X$ defined by

$$G(x,z) = \left\{ y \in \Gamma(x,z) : v(x,z) = F(x,y,z) + \beta \int f(\phi(x,y,z')z')Q(z,dz') \right\}$$

is nonempty, compact-valued and u.h.c.

Theorem 9.7' *Let* $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$, Q,Γ,F,β *and* ϕ *satisfy Assumptions 9.4-9.9 and 9.16-9.17; let* ϕ *be nondecreasing in each of the first* l *arguments, and let* v *be a unique fixed point of the operator defined in Theorem 9.6'. Then for each* $z \in Z$, $v(\cdot,z) : X \to \mathbf{R}$ *is strictly increasing.*

Theorem 9.8' *Let* $(X,\mathscr{X})$, $(Y,\mathscr{Y})$, $(Z,\mathscr{Z})$, Q,Γ,F,β *and* ϕ *satisfy Assumptions 9.4-9.7, 9.10-9.11, 9.16-9.17; let* $\phi(\cdot,\cdot,z')$ *be concave function for each* $z' \in Z$*, and let* v *be a unique fixed point of the operator defined in Theorem 9.6'. Then for each* $z \in Z$, $v(\cdot,z) : X \to \mathbf{R}$ *is strictly concave and* $G(\cdot,z) : X \to X$ *is a continuous (single-valued) function.*

References

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