Economics 8106
Macroeconomic Theory
Recitation 3

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November 15th, 2016

Outline:

- Job Search Problems
- Characterizing the Reservation Wage

1 Job Search Decision Theory

This week, we are going to be analyzing a basic model of an unemployed worker who receives routine job opportunities and decides whether or not to accept the job. There are a few ways to go about solving this problem. In his class notes, Jordan follows the method outlined by Randy Wright. You can find Professor Wright's notes on the subject among his teaching materials which I have linked from my website.

I am not going to follow this methodology. My method is taken mostly from Ljungqvist and Sargent chapter 6. The basic outlines of the methodologies are the same, but I find this method to be a little more direct and easier to follow. In his notes, Jordan goes through a very similar problem and I encourage you to read his treatment as well.

In this environment, we are considering the decision of an unemployed worker who receives a wage offer from a distribution F with bounds $[0, \bar{w}]$. The worker gets utility from her wage as simply $u(w) = w$. If the worker rejects the wage, she gets unemployment benefits of b. If the worker accepts the wage, she receives the wage that period. With probability δ, she will remain employed in the following period and probability $1 - \delta$ she will become unemployed. If she rejects the wage, she receives unemployment benefits and can search again the following period. Fired workers receive unemployment benefits and cannot search for a new job in the period they are fired.

This problem can be written as a dynamic program.

$$V(w) = \max_{accept,\ reject} \begin{bmatrix} w + \beta(\delta V(w) + (1-\delta)[b + \beta \int_0^{\bar{w}} V(w')dF(w')]), \\ b + \beta \int_0^{\bar{w}} V(w')dF(w') \end{bmatrix} \tag{1}$$

1.1 Solving the Functional Equation

The first step to solving any dynamic program is to show (if possible) that the functional equation represents a contractions and that we have a unique solution V. Define the mapping T as

$$Th(w) = \max_{accept,\ reject} \begin{bmatrix} w + \beta[\delta h(w) + (1-\delta)[b + \beta \int_0^{\bar{w}} h(w')dF(w')]], \\ b + \beta \int_0^{\bar{w}} h(w')dF(w') \end{bmatrix} \tag{2}$$

Claim 1.1. *The mapping T, as defined by (2), maps the set of bounded continuous functions into the set of bounded continuous functions, i.e. $T : B(X) \to B(X)$.*

Proof. This is evident from observing that the maximum between a constant and a continuous bounded function is a continuous and bounded function. □

Claim 1.2. *The mapping $T : B(X) \to B(X)$, as defined by (2), is a contraction.*

Proof. We will do this by showing that T satisfies Blackwell's Sufficient Conditions.

(Mono) Consider $h, g \in B(X)$ with $h(w) \geq g(w)\ \forall w \in X$. Let me define, analogously for h and g,

$$h^1(w) = w + \beta[\delta h(w) + (1-\delta)(b + \beta \int_0^{\bar{w}} h(w')dF(w'))]$$
$$h^2(w) = b + \beta \int_0^{\bar{w}} h(w')dF(w')$$

Note then that $h^1(w) \geq g^1(w)$ and $h^2(w) \geq g^2(w)$. Thus

$$Th(w) = \max_{accept,\ reject} \left[h^1(w),\ h^2(w)\right]$$
$$Th(w) \geq \max_{accept,\ reject} \left[g^1(w),\ g^2(w)\right]$$
$$Th(w) \geq Tg(w)\ \forall w \in X$$

(Disc)

$$\begin{aligned}
T(h+a)(w) &= \max_{accept,\ reject}\left[\begin{array}{l} w+\beta[\delta(h(w)+a)+(1-\delta)(b+\beta\int_0^{\bar{w}}(h(w')+a)dF(w'))], \\ b+\beta\int_0^{\bar{w}}(h(w')+a)dF(w') \end{array}\right] \\
T(h+a)(w) &= \max_{accept,\ reject}\left[\begin{array}{l} w+\beta[\delta h(w)+(1-\delta)(b+\beta\int_0^{\bar{w}} h(w')dF(w'))]+ \\ \beta(\delta+(1-\delta)\beta)a, \\ b+\beta\int_0^{\bar{w}} h(w')dF(w')+\beta a \end{array}\right] \\
T(h+a)(w) &\leq Th(w)+\beta a
\end{aligned}$$

Thus, T satisfies Blackwell's Sufficient Conditions and T is a contraction. □

Now we know that there is some solution V to the equation we defined in (1), and that solution is unique. We want to show that the optimal policy is going to take the form of a reservation wage. That is, the worker will accept any wage above some threshold w^* and reject any wage below w^*. We will do this by demonstrating that V is weakly increasing, and the reservation wage property will immediately follow. This is sometimes phrased as a "guess and verify" proof, but really it uses the corollary of the Contraction Mapping Theorem. In much of the following work, it will help to define the constant

$$V^U = b+\beta\int_0^{\bar{w}} V(w')dF(w')$$

Claim 1.3. *The solution to the functional equation (1) is weakly increasing and the optimal policy takes the form of a reservation wage.*

Proof. Suppose V is weakly increasing in w. Define w^* as

$$w^*+\beta[\delta V(w^*)+(1-\delta)V^U]=V^U$$

Then, for all $w\geq w^*$,

$$w+\beta[\delta V(w)+(1-\delta)V^U]\geq V^U$$

Thus, it must be that, for $w\geq w^*$

$$TV(w)=w+\beta[\delta V(w)+(1-\delta)V^U]\geq V^U \tag{3}$$

Therefore, I've shown that TV is (strictly) increasing for $w\geq w^*$. For $w\leq w^*$,

$$w+\beta[\delta V(w)+(1-\delta)V^U]\leq V^U$$

Thus, for $w\leq w^*$, $TV(w)=V^U$, a constant. And for $w\geq w^*$, $TV(w)\geq V^U$, thus TV is weakly increasing. By the corollary to the contraction mapping theorem, this means that the solution to (1) must be a weakly increasing function. And as we have already shown in this proof, when V is weakly increasing, the optimal policy will take the form of a reservation wage. □

With this result, we have

$$V(w) = \begin{cases} w + \beta[\delta V(w) + (1-\delta)V^U] & \text{if } w \geq w^* \\ V^U = b + \beta \int_0^{\bar{w}} V(w')dF(w') & \text{if } w < w^* \end{cases} \tag{4}$$

1.2 The Reservation Wage

In (4) we have three "unknowns," $V(w)$, V^U, and w^*. Our object of interest is the reservation wage w^*, but we will need to get the other unknowns out of the way first. We will do this by exploiting three basic relationships. And after each step we get a slightly sharper characterization. The first step is to get rid of $V(w)$, and we do this using

$$\text{if } w \geq w^*, \quad V(w) = w + \beta[\delta V(w) + (1-\delta)V^U]$$

Using this relationship, we can solve for $V(w)$ for any $w \geq w^*$.

$$\begin{aligned} V(w) &= w + \beta[\delta V(w) + (1-\delta)V^U] \\ V(w)(1-\beta\delta) &= w + \beta(1-\delta)V^U \\ V(w) &= \frac{w + \beta(1-\delta)V^U}{1-\beta\delta} \end{aligned}$$

Now we can write

$$V(w) = \begin{cases} \dfrac{w + \beta(1-\delta)V^U}{1-\beta\delta} & \text{if } w \geq w^* \\ V^U = b + \beta \int_0^{\bar{w}} V(w')dF(w') & \text{if } w < w^* \end{cases} \tag{5}$$

We also now have a clue as to how to evaluate the expectation term in V^U, but only for values greater than w^*. In order to get the other side, we can exploit the fact that at the reservation wage, the two expressions in (5) are equal.

$$\begin{aligned} \frac{w^* + \beta(1-\delta)V^U}{1-\beta\delta} &= V^U \\ w^* = (1-\beta\delta)V^U - \beta(1-\delta)V^U & \\ w^* = (1-\beta\delta-\beta+\beta\delta)V^U & \\ \frac{w^*}{1-\beta} &= V^U \end{aligned}$$

The final piece of the puzzle is to characterize the reservation wage. We can now do this using the equation $V^U = b + \beta \int_0^{\bar{w}} V(w')dF(w')$. This tells us

$$\begin{aligned} \frac{w^*}{1-\beta} &= b + \beta \int_0^{\bar{w}} V(w)dF(w) \\ \frac{w^*}{1-\beta} = b + \beta\big(F(w^*)\frac{w^*}{1-\beta}\big) &+ \frac{\beta}{1-\delta\beta}\int_{w^*}^{\bar{w}} w + \beta(1-\delta)\frac{w^*}{1-\beta}dF(w) \end{aligned}$$

$$\frac{w^*}{1-\beta}\big[1-\beta F(w^*)-\frac{\beta^2(1-\delta)(1-F(w^*))}{1-\delta\beta}\big] = b+\frac{\beta}{1-\delta\beta}\int_{w^*}^{\bar{w}} wdF(w)$$

After a great deal of algebra, we arrive at

$$\begin{aligned}
\frac{w^*}{1-\delta\beta}\big[1-\delta\beta+\beta(1-F(w^*))\big] &= b+\frac{\beta}{1-\delta\beta}\int_{w^*}^{\bar{w}} wdF(w) \\
w^* = b+\frac{\beta}{1-\delta\beta}\int_{w^*}^{\bar{w}} wdF(w) &- \frac{\beta}{1-\delta\beta}w^*(1-F(w^*)) \\
w^* &= b+\frac{\beta}{1-\delta\beta}\int_{w^*}^{\bar{w}} (w-w^*)dF(w)
\end{aligned}$$

We can get an alternative characterization which will help analyzing certain situations.

$$\begin{aligned}
w^* &= b+\frac{\beta}{1-\delta\beta}\Big(E[w-w^*]-\int_0^{w^*}(w-w^*)dF(w)\Big) \\
w^*(1+\frac{\beta}{1-\delta\beta}) &= b+\frac{\beta}{1-\delta\beta}\Big(E[w]-\int_0^{w^*}(w-w^*)dF(w)\Big) \\
w^*(1+\frac{\beta}{1-\delta\beta}) &= b+\frac{\beta}{1-\delta\beta}\Big(E[w]+\int_0^{w^*}F(w)dw\Big)
\end{aligned}$$

Now consider a new wage distribution G which has the same mean and support, but F second-order stochastic dominates G. One case of such a new distribution is a mean-preserving spread over F. G has the property that, for all $y \in [0, \bar{w}]$,

$$\int_0^y G(w)dw > \int_0^y F(w)dw$$

Then we have

$$w^*(1+\frac{\beta}{1-\delta\beta}) < b+\frac{\beta}{1-\delta\beta}\Big(E[w]+\int_0^{w^*}G(w)dw\Big)$$

Since the right-hand side is increasing by less than the left hand side (an exercise for the reader), this means that a mean-preserving spread will increase the reservation wage.