Macroeconomic Theory (ECON 8105)

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Midterm Exam

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Question 1: (55 points)

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma) \log l_t \right)$$

where $0 < \beta < 1$ and $0 < \gamma < 1$. The consumer is endowed with 1 unit of time each period, some of which can be consumed as leisure l_t, and some of which is supplied as labor n_t. The consumer is also endowed with $\bar{k}_0 > 0$ units of capital in period 0. Feasible allocations satisfy

$$c_t + k_{t+1} \leq A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t$$

where $A > 0$, $0 < \delta < 1$, and $0 < \alpha < 1$.

(a) (5 pts) Write down the social planner's problem of maximizing the representative consumer's utility subject to feasibility conditions.

Social planner's problem is

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma) \log l_t \right) \\ s.t. \quad & c_t + k_{t+1} \leq A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t \\ & c_t, k_{t+1}, l_t, n_t \geq 0 \\ & l_t + n_t \leq 1 \\ & \bar{k}_0 > 0 \text{ given} \end{aligned}$$

(b) (10 pts) Write down the Euler's equation for consumption, the intratemporal marginal rates of substitution between consumption and labor, and the transversality condition. Express each equation in terms of allocations (i.e. do not include Lagrange multipliers or prices).

Note that since utility is strictly increasing in consumption and leisure, we will have $n_t + l_t = 1$, therefore we can substitute l_t for $1 - n_t$. Let λ_t be the lagrange multiplier on the feasibility constraint. The first order conditions for the social planner's problem are given by,

$$\beta^t \frac{\gamma}{c_t} = \lambda_t \tag{1}$$

$$\beta^t \frac{1-\gamma}{1-n_t} = \lambda_t A(1-\alpha)k_t^{\alpha} n_t^{-\alpha} \tag{2}$$

$$\lambda_t = \lambda_{t+1}(A\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta)) \tag{3}$$

To find the Euler Equation (intertemporal rate of substitution), combine (1) and (3):

$$\frac{\beta^t \frac{\gamma}{c_t}}{\beta^{t+1}\frac{\gamma}{c_{t+1}}} = \frac{\lambda_t}{\lambda_{t+1}}$$

$$\frac{c_{t+1}}{c_t} = \beta \frac{\lambda_t}{\lambda_{t+1}}$$

$$\frac{c_{t+1}}{c_t} = \beta[A\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta)]$$

To find the intratemporal rate of substitution between consumption and labor, combine (1) and (2):

$$\frac{\beta^t \frac{1-\gamma}{1-n_t}}{\beta^t \frac{\gamma}{c_t}} = \frac{\lambda_t A(1-\alpha)k_t^{\alpha} n_t^{-\alpha}}{\lambda_t}$$

$$\frac{(1-\gamma)c_t}{\gamma(1-n_t)} = A(1-\alpha)k_t^{\alpha} n_t^{-\alpha}$$

The transversality condition states that the present value of the capital stock in the limit must be equal to 0. There are several equivalent ways to express this condition.

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$

$$\lim_{t\to\infty} \frac{\gamma\beta^t}{c_t} k_{t+1} = 0$$

$$\lim_{t\to\infty} \frac{\gamma\beta^t}{c_t}[A\alpha k_t^{\alpha-1} n_t^{1-\alpha} + (1-\delta)]k_t = 0$$

(c) (5 pts) Define an Arrow-Debreu equilibrium for this economy.

An **Arrow-Debreu Equilibrium** is

- an allocation for the HH: $z^H = \{(c_t, l_t, n_t, k_{t+1})\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t, w_t, r_t)\}_{t=0}^{\infty}$

such that

(HH) Given p, z^H solves

$$\max_{c_t,l_t,n_t,k_t} \sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma)\log l_t\right)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\delta)k_t\right] \leq \sum_{t=0}^{\infty} \left[w_t n_t + r_t k_t\right]$$

$$l_t + n_t \leq 1,\ \forall t$$

$$c_t, k_{t+1}, l_t, n_t \geq 0,\ \forall t$$

$$k_0 > 0, \quad \text{given}$$

(Firm) Given p, z^F solves

$$\max_{y_t^f, k_t^f, n_t^f} \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right]$$

$$s.t.$$

$$y_t^f \leq A k_t^{\alpha} n_t^{1-\alpha},\ \forall t$$

$$k_t^f, n_t^f, y_t^f \geq 0, \forall t$$

(Mkt) For all t,

$$\text{(Goods Market)} \quad c_t + k_{t+1} = y_t^f + (1-\delta)k_t$$

$$\text{(Labor Market)} \quad n_t = n_t^f$$

$$\text{(Capital Market)} \quad k_t = k_t^f$$

(d) (10 pts) Carefully explain how the objects in part (c) can be used to construct a sequential markets equilibrium.

Let (Z,p) form an Arrow Debreu equilibrium as defined in part (c), where $Z = (z^H, z^f)$, $z^H = \{(c_t, l_t, n_t, k_{t+1})\}_{t=0}^{\infty}$, $z^F = \{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}$, and $p = \{(p_t, w_t, r_t)\}_{t=0}^{\infty}$. Define, for all time t,

$$\hat{c}_t = c_t \qquad \text{(HH)}$$

$$\hat{l}_t = l_t$$

$$\hat{n}_t = n_t$$

$$\hat{k}_{t+1} = k_{t+1}$$

$$\hat{y}_t^f = y_t^f \qquad \text{(Firms)}$$

$$\hat{k}_t^f = k_t^f$$

$$\hat{n}_t^f = n_t^f$$

$$\hat{r}_t^k = \frac{r_t}{p_t} \qquad \text{(Prices)}$$

$$\hat{w}_t = \frac{w_t}{p_t}$$

$$\hat{r}_t^b = \frac{p_t}{p_{t+1}} - 1$$

Also, allow me to define a sequence of bonds $\{\hat{b}_{t+1}\}_{t=0}^{\infty}$,

$$\hat{b}_1 = \hat{w}_0\hat{n}_0 + \hat{r}_0^k\hat{k}_0 - \hat{c}_0 - \hat{k}_1 + (1-\delta)\hat{k}_0$$
$$\hat{b}_{t+1} = \hat{w}_0\hat{n}_0 + \hat{r}_0^k\hat{k}_0 - \hat{c}_0 - \hat{k}_1 + (1-\delta)\hat{k}_0 + (1+\hat{r}_t^b)\hat{b}_t$$

and a large debt limit B such that for all t, $\hat{b}_{t+1} \geq -B$. Then $(\hat{Z}, \hat{p})$ form a Sequential Markets equilibrium where $\hat{Z} = (\hat{z}^H, \hat{z}^f)$, $\hat{z}^H = \{(\hat{c}_t, \hat{l}_t, \hat{n}_t, \hat{k}_{t+1}, \hat{b}_{t+1})\}_{t=0}^{\infty}$, $\hat{z}^F = \{(\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f)\}_{t=0}^{\infty}$, and $\hat{p} = \{(\hat{r}_t^b, \hat{w}_t, \hat{r}_t^k)\}_{t=0}^{\infty}$.

(e) (15 pts) Define Pareto Optimality in this economy. State and prove the First Welfare Theorem.

Our economy has only one agent, so the definitions and proofs can be modified slightly.

An allocation z is <u>Pareto Optimal</u> if it is feasible, and there exists no other feasible allocation $\hat{z}$ such that

$$\sum_{t=0}^{\infty} \beta^t \left(\gamma \log \hat{c}_t + (1-\gamma)\log \hat{l}_t\right) > \sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma)\log l_t\right)$$

First Welfare Theorem: Let $\mathcal{E}$ be a production economy such that $\forall i$, $k_0^i > 0$ and U^i is strictly increasing. If (z,p) is an Arrow-Debreu (competitive) equilibrium (where $z = \left\{\left(z^{H,i}\right)_{i\in I}, z^F\right\}$), then z is Pareto optimal.

Proof. Suppose, by contradiction, that z is not Pareto optimal. By definition of Arrow-Debreu equilibrium, the allocation z is feasibly. Then it must be that there exists another feasible allocation $\hat{z}$ such that

$$\sum_{t=0}^{\infty} \beta^t \left(\gamma \log \hat{c}_t + (1-\gamma)\log \hat{l}_t\right) > \sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma)\log l_t\right)$$

Claim $\sum_{t=0}^{\infty} p_t\left[\hat{c}_t + \hat{x}_t\right] > \sum_{t=0}^{\infty}\left[w_t\hat{n}_t + r_t\hat{k}_t\right]$

Suppose not, i.e. $\sum_{t=0}^{\infty} p_t\left[\hat{c}_t + \hat{x}_t\right] \leq \sum_{t=0}^{\infty}\left[w_t\hat{n}_t + r_t\hat{k}_t\right] + \pi$

Then we have that $\hat{z}^H$ satisfies budget constraint and yields higher utility, which contradicts z^H being part of the Arrow-Debreu equilibrium.

Thus, we have

$$\sum_{t=0}^{\infty} p_t\left[\hat{c}_t + \hat{x}_t\right] > \sum_{t=0}^{\infty}\left[w_t\hat{n}_t + r_t\hat{k}_t\right]$$

Note that, since the firm is CRS, and $k_t = k_t^f,\ n_t = n_t^f$,

$$p_t y_t = p_t A k_t^{\alpha} n_t^{1-\alpha} = r_t k_t + w_t n_t$$
$$\sum_{t=0}^{\infty} p_t y_t = \sum_{t=0}^{\infty} p_t A k_t^{\alpha} n_t^{1-\alpha} = \sum_{t=0}^{\infty} r_t k_t + w_t n_t$$

Substituting in this condition, we get,

$$\sum_{t=0}^{\infty} p_t \left[\hat{c}_t + \hat{x}_t\right] > \sum_{t=0}^{\infty} \left[p_t A \hat{k}_t^{\alpha} \hat{n}_t^{1-\alpha}\right]$$

Subtituting in the profit condition and the firm's problem, we have

Note that by contradiction hypothesis, $\hat{z}$ is feasible.

$$\hat{c}_t^i + \hat{x}_t^i \leq A \hat{k}_t^{\alpha} \hat{n}_t^{1-\alpha}$$

Multiplying both sides by p_t and summing across of time, we have

$$\sum_{t=0}^{\infty} p_t \left[\hat{c}_t + \hat{x}_t\right] \leq \sum_{t=0}^{\infty} \left[p_t A \hat{k}_t^{\alpha} \hat{n}_t^{1-\alpha}\right]$$

which is a contradiction. □

(f) (10 pts) State the Second Welfare Theorem and explain its implications for this economy.

The statement of the Second Welfare Theorem is in the second week recitation notes. The utility function is strictly increasing and concave, so we know that the Second Welfare Theorem applies to this economy. Furthermore, since there is only one agent in the economy, we know that there are no feasible redistributions of endowments of initial capital stock. Therefore, it must be that the Pareto Optimal solution set to the social planners problem in part (a) is the same as the set of solution allocations to the AD equilibrium defined in part (c).

Question 2: (45 points)

Consider an infinite horizon economy with an infinitely lived representative agent. There is a single commodity that can be consumed or invested in capital. Capital fully depreciates from one period to another. The agent supplies 1 unit of labor inelastically. The agent's preferences are of the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$. The aggregate resource constraint is

$$c_t + k_{t+1} = F(k_t, 1)$$

where c_t denotes period t aggregate consumption, and k_t the aggregate capital stock. Assume that both u is differentiable, strictly concave, strictly increasing and satisfies the Inada conditions. Assume that F is differentiable, strictly concave, strictly increasing, satisfies the Inada conditions in k, and has constant returns to scale.

(a) (5 pts) Write the social planners problem as a dynamic programming problem for capital stock.

Let $f(k) = F(K, 1)$. The social planners problem can be written recursively as

$$v(k) = \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta v(k')$$
$$\Gamma(k) = \{k' \in \mathbb{R}_+ | \ 0 \le k' \le f(k)\}$$

(b) (10 pts) State any properties of the value function that you can infer from characteristics of u and F, and for each property, explain which characteristics of u and F allow you to do so.

First note that since f satisfies the Inada conditions, there is a maximum sustainable level of capital, and since f and u and both continuous, both functions are bounded. This allows us to use theorems from Chapter 4.2 of SLP. Since u and f are strictly increasing, We know that v is strictly increasing. Since u and F are strictly concave, We know that v is strictly concave. Since u and f are continuously differentiable and strictly concave, We know that v is continuously differentiable.

(c) (5 pts) Write the first order condition and the envelope condition.

$$u'(f(k) - k') = \beta v'(k') \tag{FOC}$$
$$v'(k) = u'(f(k) - k')f'(k) \tag{ENV}$$

(d) (10 pts) Define a steady state for this economy. Prove that such a steady state exists and is unique. Call this steady state level of capital, k^*.

A steady state has $k = k'$. Thus,

$$u'(f(k) - k) = \beta v'(k)$$
$$u'(f(k) - k) = \beta u'(f(k) - k)f'(k)$$
$$\frac{1}{\beta} = f'(k)$$

Since $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$, and $f'(k)$ is a continuous function, there must exist some k^* such that $f'(k^*) = \frac{1}{\beta}$, by the intermediate value theorem. Moreover, since f is strictly concave, $f'(\cdot)$ is strictly decreasing. Thus, k^* must be unique.

(e) (15 pts) Suppose that policy function $g_k(k)$ is the solution the recursive problem defined in part (a). Use the first order condition and the envelope condition to show that $g_k(k)$ is strictly increasing.

Suppose for a contradiction that there for some $k_1 > k_2$, $g_k(k_1) \leq g_k(k_2)$. Then, since f is strictly increasing,

$$f(k_1) - g_k(k_1) > f(k_2) - g_k(k_2)$$

Since u is strictly concave,

$$\begin{aligned} u'(f(k_1) - g_k(k_1)) &< u'(f(k_2) - g_k(k_2)) \\ \beta v'(g_k(k_1)) &< \beta v'(g_k(k_2)) \end{aligned}$$

Since v is strictly concave, this implies that

$$\begin{aligned} v'(g_k(k_1)) &< v'(g_k(k_2)) \\ g_k(k_1) &> g_k(k_2) \end{aligned}$$

This is a contradiction. Thus, it must be that $g_k(k_1) > g_k(k_2)$.