Macroeconomic Theory (ECON 8105)

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Final Exam

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There are two questions on two pages for a total of 100 points. Write down the answers in the provided booklet. You have 1 hour and 15 minutes. Do your best and don't worry about the grades. Good luck!

Question 1: (50 points)

Consider an infinite horizon setting in which there is a representative consumer and a representative firm as in the standard single sector growth model. The utility function of the representative consumer is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t, g_t)$$

where g_t is the amount of government goods and services produced in each period.

The feasibility constraint for the firm is:

$$c_t + g_t + x_t \leq F(k_t, n_t)$$

There is no technological change. Investment is done at the household level, and the standard law of motion for capital is assumed to hold.

Suppose that the government can freely borrow and lend (i.e, it faces a present value budget constraint), and can levy taxes on labor and capital income in order to finance expenditures. Assume that the consumer takes g_t, τ_{kt}, τ_{nt} as given when making its decisions.

(a) (10 pts) Define a Tax Distorted Competitive Equilibrium for this economy. A **Tax Distorted Competitive Equilibrium** is

- an allocation for the HH: $z^H = \{(c_t, l_t, n_t, k_t, x_t)\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t, w_t, r_t)\}_{t=0}^{\infty}$
- a government policy: $g = \{(g_t, \tau_{kt}, \tau_{nt})\}_{t=0}^{\infty}$

such that

(HH) Given p and g, z^H solves

$$\max_{c_t,l_t,n_t,k_t,x_t} \quad \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, g_t)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t c_t + p_t x_t \leq \sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t + r_t(1-\tau_{kt})k_t$$

$$k_{t+1} \leq x_t + (1-\delta)k_t, \ \forall t$$

$$l_t + n_t \leq 1, \ \forall t$$

$$c_t, k_{t+1}, l_t, n_t \geq 0, \ \forall t$$

$$k_0 > 0, \quad \text{given}$$

(Firm) Given p, z^F solves

$$\max_{\{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f \right]$$

$$s.t.$$

$$y_t^f \leq F(k_t^f, n_t^f), \ \forall t$$

$$k_t^f, n_t^f, y_t^f \geq 0, \forall t$$

(Mkt) For all t,

(Goods Market) $c_t + x_t + g_t = y_t^f \leq F(k_t^f, n_t^f)$

(Labor Market) $n_t = n_t^f$

(Capital Market) $k_t = k_t^f$

(Govt)

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} w_t \tau_{nt} n_t + r_t \tau_{kt} k_t$$

(b) (10 pts) Assume an interior solution, and write down the First Order Conditions of the equilibrium you defined in Part (a).

$$\beta^t u_c(c_t, 1-n_t, g_t) = \lambda p_t \tag{c_t}$$

$$\beta^t u_l(c_t, 1-n_t, g_t) = \lambda w_t(1-\tau_{nt}) \tag{n_t}$$

$$p_t = r_{t+1}(1-\tau_{kt+1}) + p_{t+1}(1-\delta) \tag{k_{t+1}}$$

$$r_t = p_t F_k(k_t, n_t) \tag{k_t^f}$$

$$w_t = p_t F_n(k_t, n_t) \tag{n_t^f}$$

$$\sum_{t=0}^{\infty} p_t c_t + p_t x_t = \sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t + r_t(1-\tau_{kt})k_t \tag{λ_t}$$

These equations are all the first order conditions of the maximization problems, but do not fully characterize equilibrium. A full characterization will also include the equilibrium market clearing conditions and the government budget constraint. As you showed in the problem set, you do not need to include the government's budget if you have the household budget and the resource constraint.

(c) (15 pts) Derive the Implementability Constraint and formulate the Ramsey problem when the government maximizes a weighted sum of the consumer's discounted lifetime utility. The implementability constraint can be derived by...

$$\beta^t \frac{u_c(t)}{u_c(0)} p_0 = p_t$$

$$\beta^t \frac{u_l(t)}{u_c(0)} p_0 = w_t(1-\tau_{nt})$$

$$\sum_{t=0}^{\infty} p_t c_t + p_t(k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} r_t(1-\tau_t)k_t + w_t(1-\tau_{nt})n_t$$

$$\sum_{t=0}^{\infty} p_t c_t - w_t(1-\tau_{nt})n_t \leq \sum_{t=0}^{\infty} r_t(1-\tau_t)k_t + p_t(1-\delta)k_t - p_t k_{t+1}$$

$$\sum_{t=0}^{\infty} p_t c_t - w_t(1-\tau_{nt})n_t \leq$$

$$(r_0(1-\tau_0) + p_0(1-\delta))k_0 + \sum_{t=0}^{\infty}(r_{t+1}(1-\tau_{t+1}) + (1-\delta)p_{t+1})k_{t+1} - p_t k_{t+1}$$

$$\sum_{t=0}^{\infty} p_t c_t - w_t(1-\tau_{nt})n_t \leq (r_0(1-\tau_{k0}) + p_0(1-\delta))k_0 + \lim_{T\to\infty} p_T k_{T+1}$$

$$\sum_{t=0}^{\infty} p_t c_t - w_t(1-\tau_{nt})n_t \leq (r_0(1-\tau_{k0}) + p_0(1-\delta))k_0$$

$$\sum_{t=0}^{\infty} \beta^t \frac{p_0}{u_c(0)}(u_c(t)c_t - u_l(t)n_t) \leq (p_0 F_k(0)(1-\tau_{k0}) + p_0(1-\delta))k_0$$

$$\sum_{t=0}^{\infty} \beta^t (u_c(t)c_t - u_l(t)n_t) \leq u_c(0)(F_k(0)(1-\tau_{k0}) + (1-\delta))k_0 \qquad \text{(IC)}$$

The Ramsey Problem is

$$\max_{c_t, k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t, g_t)$$

$$\begin{aligned}
&\text{such that} \\
&\text{(IC)} && (1) \\
c_t + k_{t+1} - (1-\delta)k_t + g_t &\leq F(k_t, n_t) && (2) \\
c_t,\ k_{t+1}, n_t &\geq 0 && (3) \\
n_t &\leq 1 && (4) \\
k_0 &> 0 \text{ given}
\end{aligned}$$

(d) (15 pts) If the government acts benevolently in choosing $(g_t, \tau_{nt}, \tau_{kt})$, will it be true that $\tau_{kt} \to 0$? That is, does the Chamley-Judd characterization of the asymptotic behavior of Ramsey tax systems extend to this setting in which g_t enters the utility function? Prove your answer. Assume that in the Ramsey allocation, all quantities converge to constant levels, $c_t \to c_\infty$, etc.

$$\mathcal{L} = \sum_{t=1}^{\infty} \left(\beta^t W(c_t, n_t, g_t) + \mu_t (F(k_t, n_t) - c_t - k_{t+1} - (1-\delta)k_t - g_t) \right) + W_0$$

where

$$W(c_t, n_t, g_t) = u(c_t, 1-n_t, g_t) + \lambda(u_c(t)c_t - u_l(t)n_t)$$

The first order conditions are

$$\begin{aligned}
\beta^t W_c(t) &= \mu_t \\
\beta^t W_n(t) &= -\mu_t Fn(t) \\
\mu_t &= \mu_{t+1} F_k(t+1) + \mu_{t+1}(1-\delta)
\end{aligned}$$

Thus, we have

$$\frac{W_c(t)}{W_c(t+1)} = F_k(t+1) + (1-\delta)$$

In the limit, if we have steady state, then we know

$$1 = F_k(k_{rp}, n_{rp}) + (1-\delta)$$

From the TDCE, we have

$$1 = F_k(k_\infty, n_\infty)(1-\tau_{k\infty}) + (1-\delta)$$

Thus, in order to implement the Ramsey allocation in the limit, we must have $\tau_{kt} \to 0$, i.e. the Chamley Judd result still holds in this context.

Question 2: (50 points)

Consider an infinite horizon, representative agent model in which output is produced in each period according to the production function of the Ak form, where $A > 0$ and k is the capital stock. There is full depreciation. The consumer's preference is of the form:

$$\sum_t \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where $\sigma > 1$

The government levies a capital tax rate τ_t in each period to finance an exogenous stream of expenditures $\{g_t\}$. Government balances budget every period. Assume that this tax rate is i.i.d over time and can take one of two values with:

$$Pr(\tau_t = \bar{\tau} - \epsilon) = Pr(\tau_t = \bar{\tau} + \epsilon) = \frac{1}{2}$$

where $\epsilon > 0$ and that $0 < \bar{\tau} - \epsilon < \bar{\tau} + \epsilon < 1$.

(a) (10 pts) Define a Stochastic Tax Distorted Competitive Equilibrium. A **Tax Distorted Competitive Equilibrium** is

- an allocation for the HH: $z^H = \{(c_t(s^t), k_{t+1}(s^t))\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f(s^t), k_t^f(s^t))\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t(s^t), r_t(s^t))\}_{t=0}^{\infty}$

such that

(HH) Given p, $\forall i \in I$, z^H solves

$$\max_{(c_t(s^t), k_{t+1}(s^t), x_t(s^t), n_t(s^t), l_t(s^t))} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \frac{c_t(s^t)^{1-\sigma}}{1-\sigma}$$

$$s.t.$$

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) \left[c_t(s^t) + k_{t+1}(s^t)\right] \leq \sum_{t=0}^{\infty} \sum_{s^t} r_t(s^t) k_t(s^{t-1})(1 - \tau(s_t))$$

$$c_t(s^t), k_{t+1}(s^t) \geq 0, \ \forall t, \ s^t$$

$$k_0, \ s_0 \ > 0, \quad \text{given}$$

(Firm) Given p, z^F solves

$$\max_{(y_t^f(s^t), k_t^f(s^t), n_t^f(s^t))} \sum_{t=0}^{\infty} \sum_{s^t} \left[p_t(s^t) y_t^f(s^t) - r_t(s^t) k_t^f(s^t)\right]$$

$$s.t.$$

$$y_t^f(s^t) \leq A k_t^f(s^t), \ \forall t, \ s^t$$

$$k_t^f(s^t), y_t^f(s^t) \geq 0, \forall t$$

(Mkt) For all t, s^t,

$$\begin{aligned} \text{(Goods Market)} \quad & c_t(s^t) + k_{t+1}(s^t) + g_t(s^t) = y_t^f(s^t) \\ \text{(Capital Market)} \quad & k_t(s^{t-1}) = k_t^f(s^t) \end{aligned}$$

(b) (10 pts) Assume an interior solution. Write down the equations that fully characterize a solution to the equilibrium.

$$c_t(s^t)^{-\sigma} = \beta E[A(1-\tau(s_{t+1}))c_{t+1}(s^{t+1})^{-\sigma}|s^t] \tag{5}$$
$$c_t(s^t) + k_{t+1}(s^t) + g_t(s^t) = Ak_t(s^t) \tag{6}$$
$$g_t(s^t) = \tau(s_t)r_t(s^t)k_t(s^t) \tag{7}$$
$$\lim_{T\to\infty} \beta^T(c_T(s^T))^{-\sigma}k_{T+1}(s^T) = 0 \text{ a.e.} \tag{8}$$

(c) (5 pts) The solution to this problem can also be obtained by solving a problem. Clearly and carefully lay out the associated Bellman equation of this planning problem. You do not need to prove that they are equivalent.

$$V(k,s) = \max_{k' \in \Gamma(k,s)} \frac{(A(1-\tau(s))k - k')^{1-\sigma}}{1-\sigma} + \beta E[V(k',s')] \tag{9}$$
$$\Gamma(k,s) = \{k' | 0 \le k' \le A(1-\tau(s))\} \tag{10}$$

(d) (15 pts) Prove that the consumption-to-output ratio is constant and decreasing with ϵ. You may use propositions that describe homogeneity properties of value function and policy functions with out proof if you state them clearly and briefly justify their use.

You can see Week 7 recitation notes for a proof of the characterization of policy functions that are constant fractions of output.

$$g_k(k,s) = \phi A(1-\tau(s))k$$
$$g_c(k,s) = (1-\phi)A(1-\tau(s))k$$

Then, the Euler equation becomes

$$1 = \beta E[A(1-\tau(s_{t+1}))\big(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\big)^{-\sigma}|s^t]$$
$$1 = \beta E[A(1-\tau(s_{t+1}))\big(\frac{(1-\phi)A(1-\tau(s_{t+1}))k_{t+1}(s^{t+1})}{(1-\phi)A(1-\tau(s_t))k_t(s^t)}\big)^{-\sigma}|s^t]$$
$$1 = \beta E[A(1-\tau(s_{t+1}))\big(\frac{(1-\phi)A(1-\tau(s_{t+1}))\phi A(1-\tau(s_t))k_t(s^t)}{(1-\phi)A(1-\tau(s_t))k_t(s^t)}\big)^{-\sigma}|s^t]$$
$$1 = \beta E[A(1-\tau(s_{t+1}))\big((1-\tau(s_{t+1}))\phi A\big)^{-\sigma}|s^t]$$
$$1 = \beta\phi^{-\sigma}A^{1-\sigma}E[(1-\tau(s_{t+1}))^{1-\sigma}] \tag{11}$$
$$\phi = (\beta A^{1-\sigma}E[(1-\tau(s_{t+1}))^{1-\sigma}])^{\frac{1}{\sigma}} \tag{12}$$

Define $g(\tau) = (1-\tau)^{1-\sigma}$. Note that if $\sigma > 1$, then g is a convex function. (This can be shown by demonstrating that the second derivative is strictly positive). The Rothschild-Stiglitz Theorem presented in class says that for some τ_1 and τ_2, where τ_2 is a mean-preserving spread over τ_1, then,

$$E[g(\tau_2)] > E[g(\tau_1)] \tag{13}$$

It can easily be shown that an increase in ϵ is equivalent to a mean-preserving spread. Let $\epsilon_2 > \epsilon_1$, $\tau_1 = \tau \pm \epsilon_1$ and $\tau_2 = \tau \pm \epsilon_2$. Then, by combining (13) and (12), we get

$$\begin{aligned} \phi_2 &> \phi_1 \\ 1 - \phi_2 &< 1 - \phi_1 \end{aligned}$$

The expected consumption to output ratio is

$$E[\frac{c_t(s^t)}{Ak_t(s^t)}] = (1-\phi)E[1-\tau(s_t)]$$

Thus, the consumption to output ratio is decreasing with an increase in ϵ.

(e) (5 pts) Concerning the result in part (d), interpret the economic meaning of $\sigma > 1$ in contrast with $0 < \sigma < 1$. Describe what it means about agent's preferences with respect to risk.

The parameter σ is a measure of risk aversion. For any $\sigma > 0$, the agent is risk averse. However, a more risk averse agent with $\sigma > 1$ will increase savings in response to more risk in future returns so as to assure a more smooth path of consumption. If $\sigma < 1$, the agent is less risk averse and will prefer to increase consumption when the return to savings becomes more risky. These are both "risk averse" reactions to risk, but lead to opposite conclusions. If $\sigma = 1$, i.e. we have log utility, then the agents savings and consumption behavior will not depend on risk. This phenomenon is sometimes described as "the income and substitution effects canceling out." It's never been made clear to me what those "effects" are precisely, but it does seem to characterize some concept of the trade-off between risky future income and consumption today.