Economics 8105
Macroeconomic Theory
Recitation 1

Conor Ryan

September 6th, 2016

Adapted From Anh Thu (Monica) Tran Xuan's Notes
Last Updated September 20th, 2016

Outline:

- Dynamic Economic Environment
- Arrow-Debreu Equilibrium
- Sequential Markets Equilibrium
- Characterizing the Arrow-Debreu Equilibrium

1 Environment with Endowments

Definition 1.1. A **pure exchange economy** is a set of commodities, $\{1, ...\ell\}$, and a set consumers, $I = \{1, \dots, n\}$. Each consumer $i \in I$ has consumption set X^i (typically $\mathbb{R}^l_+$ in a static environment), initial endowment $e^i \in X^i$, and utility $U^i : X^i \to \mathbb{R}$. We can express the economic environment as $\mathcal{E} = \{(e^i, U^i)_{i \in I}\}$.

In this recitation, we will be considering economies with the following characteristics:

- Pure exchange economy with one commodity
- Discrete time $t = 0, 1, 2, \dots$
- Infinitely lived consumers, indexed by $i \in I = \{1, 2\}$
- Each consumer i has allocation $c^i = (c^i_0, c^i_1, c^i_2, \dots)$, $c^i_t \in \mathbb{R}_+$
- Utility function: $U^i\left(c^i_0, c^i_1, c^i_2, \dots\right) = \sum_{t=0}^{\infty} \beta^t u^i(c^i_t)$, where $0 < \beta < 1$

- Endowment: $e^i = (e^i_0, e^i_1, \dots)$

2 Arrow-Debreu Equilibrium

Market Structure:

- The consumers trade the single commodity.
- Future markets are open in period 0, during which consumers trade contingent claims for all periods. No more trading occurs.

Definition 2.1. In this economy, an **Arrow-Debreu Equilibrium** is

- an allocation for HHs: $z^{H,i} = \{c^i_t\}_{t=0}^{\infty}$, $\forall i \in \{1,2\}$
- a system of prices: $p = \{p_t\}_{t=0}^{\infty}$

such that

(HH) Given p, $\forall i \in \{1,2\}$, $z^{H,i}$ solves

$$\max_{\{c^i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u^i(c^i_t)$$
$$s.t.$$
$$\sum_{t=0}^{\infty} p_t c^i_t \leq \sum_{t=0}^{\infty} p_t e^i_t \qquad (\lambda^i)$$
$$c^i_t \geq 0 \qquad (\gamma^i_t)$$

(Mkt) For all t,

(Goods Market Clears) $\sum_{i \in I} c^i_t = \sum_{i \in I} e^i_t$

3 Sequential Markets Equilibrium

Market Structure:

- The agents trade the commodity and one period bonds.
- Markets open at the beginning of each period, during which consumers trade goods and bonds for that period only.
- Consumers are constrained by a ”non-binding” debt limit.

Definition 3.1. In this economy, a **Sequential Market Equilibrium** is

- an allocation for HHs: $z^{H,i} = \{(c_t^i, b_t^i)\}_{t=0}^{\infty}, \forall i \in I$
- a system of prices: $p = \{r_t\}_{t=0}^{\infty}$

such that

(HH) Given p, $\forall i \in I$, $z^{H,i}$ solves

$$
\begin{aligned}
\max_{\{(c_t^i, b_t^i)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \\
s.t. & \\
c_0^i + b_1^i & \leq e_0^i \\
c_t^i + b_{t+1}^i & \leq e_t^i + (1 + r_t) b_t^i && (\mu_t^i) \\
c_t^i & \geq 0 && (\gamma_t^i) \\
b_t^i & \geq \underline{\mathrm{B}}
\end{aligned}
$$

(Mkt) For all t,

$$
\text{(Goods Market Clears)} \sum_{i \in I} c_t^i = \sum_{i \in I} e_t^i
$$

$$
\text{(Bonds Market Clears)} \sum_{i \in I} b_t^i = 0
$$

4 Characterizing the Arrow-Debreu Equilibrium

Assumption 4.1. *For all i, u^i is differentiable.*

Under this assumption, we can write the Lagrangian and the Kuhn-Tucker first order conditions:

$$
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) + \lambda^i \left(\sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \right) + \sum_{t=0}^{\infty} \gamma_t c_t^i
$$

FOCs:

$$
\begin{aligned}
&[c_t^i] && \beta^t u^{i'}(c_t^i) - \lambda^i p_t + \gamma_t^i = 0 \\
&[\lambda^i] && \sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \geq 0 \\
& && \lambda^i \geq 0 \\
& && \lambda^i \left(\sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \right) = 0 \\
&[\gamma_t^i] && c_t^i \geq 0 \\
& && \gamma_t^i \geq 0 \\
& && \gamma_t^i c_t^i = 0
\end{aligned}
$$

This is equivalent to

$$[c_t^i] \qquad \beta^t u^{i'}(c_t^i) - \lambda^i p_t \leq 0 \; (= 0 \text{ if } c_t^i > 0) \tag{1}$$

$$[\lambda^i] \quad \sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \geq 0 \; (= 0 \text{ if } \lambda_t^i > 0) \tag{2}$$

Assumption 4.2. *For all i, u^i is strictly increasing.*

Proposition 4.1. *Under assumption 4.2, the budget constraint is binding.*

Assumption 4.3. *For all i, u^i satisfies Inada conditions: $\lim_{c\to\infty} u^{i'}(c) = 0$ and $\lim_{c\to 0} u^{i'}(c) = \infty$.*

Proposition 4.2. *If $\lim_{c\to 0} u'(c) \to \infty$ (A4.3), then in equilibrium, $c > 0$.*

Thus, under assumptions 4.1 through 4.3, both (1) and (2) bind with equality. Note that proposition 4.2 and the Kuhn-Tucker conditions imply that $\gamma_t^i = 0, \ \forall t$. We can rewrite the FOCs as

$$[c_t^i] \qquad \beta^t u^{i'}(c_t^i) = \lambda^i p_t \tag{1'}$$

$$[\lambda^i] \quad \sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t e_t^i \tag{2'}$$

From (1') we have the Euler's Equation, or intertemporal subtitution of consumption:

$$\frac{u^{i'}(c_t^i)}{u^{i'}(c_{t+1}^i)} = \beta \frac{p_t}{p_{t+1}}$$

Question. *Are the FOCs necessary conditions for the maximization problem?*

A: Yes. Because the contraints are linear in c, they satisfy the constraint qualification of Kuhn-Tucker Theorem.

Question. *Under what additional conditions are the FOCs sufficient for the maximization problem?*

A: Utility is concave. Note that the constraints are linear, and therefore the constrained set is convex. ~~Also, under assumption 4.3 the constrained set is open~~. You can verify that all the conditions are met to apply Kuhn-Tucker under convexity. (See Sundaram for more details on the Kuhn-Tucker Theorem.)

5 Environment with Production

- Discrete time $t = 0, 1, 2, \ldots$
- Production economy with one commodity
- HHs:
  - Infinitely lived n consumers, indexed by $i \in I = \{1, \ldots, n\}$
  - Utility function: $U^i\left(\{(c_t^i, l_t^i)\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i, l_t^i)$
  - Consumers invest x_t^i
  - Consumers have capital stock k_t^i which depreciates at a rate δ
  - Law of Motion of Capital: $k_{t+1} \leq x_t + (1-\delta)k_t$
  - Consumers divide time between leisure, l_t^i, and labor, n_t^i
  - Endownment: 1 unit of time each period, initial capital k_0^i
  - HHs rent out capital and labor services to firms, receiving capital and labor income.
  - Consumers own a share of firm profits θ^i such that $\theta^i \geq 0$, $\sum_{i \in I} \theta^i = 1$
- Firms: only 1 sector producing goods that can either be consumed or invested
  - One representative firm. Final good is produced by: $y_t^f = F(k_t^f, n_t^f)$
  - Typical properties of F are increasing, concave, and homogeneous of degree one (constant returns to scale).

Definition 5.1. An **Arrow-Debreu Equilibrium** is

- an allocation for HHs: $\forall i \in I,\ z^{H,i} = \{(c_t^i, l_t^i, n_t^i, k_t^i, x_t^i)\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t, w_t, r_t)\}_{t=0}^{\infty}$

such that

(HH) Given p, $\forall i \in I$, $z^{H,i}$ solves

$$\begin{aligned}
\max_{c_t^i, l_t^i, n_t^i, k_t^i, x_t^i} \quad & \sum_{t=0}^{\infty} \beta^t u^i(c_t^i, l_t^i) \\
s.t. \\
\sum_{t=0}^{\infty} p_t \left[c_t^i + x_t^i\right] &\leq \sum_{t=0}^{\infty} \left[w_t n_t^i + r_t k_t^i\right] + \pi^i \\
k_{t+1}^i &\leq x_t^i + (1-\delta)k_t^i,\ \forall t \\
l_t^i + n_t^i &\leq 1,\ \forall t \\
c_t^i, k_{t+1}^i, l_t^i, n_t^i &\geq 0,\ \forall t \\
k_0^i &> 0, \quad \text{given}
\end{aligned}$$

(Firm) Given p, z^F solves

$$\begin{aligned}
\max_{\{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right] \\
s.t. \\
y_t^f &\leq F(k_t^f, n_t^f),\ \forall t \\
k_t^f, n_t^f, y_t^f &\geq 0, \forall t
\end{aligned}$$

(Mkt) For all t,

$$\begin{aligned}
\text{(Goods Market)} \sum_{i \in I} \left[c_t^i + x_t^i\right] &= y_t^f \leq F(k_t^f, n_t^f) \\
\text{(Labor Market)} \sum_{i \in I} n_t^i &= n_t^f \\
\text{(Capital Market)} \sum_{i \in I} k_t^i &= k_t^f
\end{aligned}$$

(Profits) $\forall i,\ \pi^i = \theta_i \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right]$